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Truncated Units

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Conditions on the coefficients of the irreducible polynomial $f(X)=X^n+k_1X^{n-1}+\cdots+k_{n-1}X+k_n$, under which $\varepsilon=1+x\omega+y\omega^2$ is a unit of the field $\mathbb{Q}(\omega)$, ω being a root of $f(X)$, are obtained. The results cover $2^{n-1}n!$ cases, one of which is the case $k_1=k_2=\cdots=k_{n-1}=0$ considered by L. Bernstein (1975, *Math. Ann.* **213**, 275–279).

1. INTRODUCTION

In [2], L. Bernstein considered a pure algebraic number field $\mathbb{Q}(\omega)$ of degree n over $\mathbb{Q}$ with $\omega^n=m\in\mathbb{N}$ and asked under what conditions $\varepsilon=e+x\omega+y\omega^2$ is a unit of $\mathbb{Q}(\omega)$. He calculated the absolute norm of ε and obtained

$$N(\varepsilon)=e^n+(-1)^{n-1}m\left(\sum_{i=0}^{n}\left[\binom{n-1-i}{i-1}+\binom{n-i}{i}\right]x^{n-2i}(-ey)^i\right)+m^2y^n. \tag{1.1}$$

This led him to take $e=1$, $y=-1/z$ with $x, z\in\mathbb{N}=\{1, 2, ...\}$, and to give a value of m for which $\varepsilon=1+x\omega-\omega^2/z$ is a unit of $\mathbb{Q}(\omega)$.

In fact, when

$$M_n=\sum_{i=0}^{n}\left[\binom{n-1-i}{i-1}+\binom{n-i}{i}\right]D^{n-2i}d^i, \qquad \omega=\sqrt[n]{M_n}>1, \tag{1.2}$$

with $D\in\mathbb{N}$, $d\in\mathbb{Z}=\{..., -2, -1, 0, 1, 2, ...\}$, $d\mid D$, L. Bernstein found that

$$\varepsilon=1+\frac{D}{d}\omega-\frac{\omega^2}{d} \tag{1.3}$$

and

$$\varepsilon'=1-\frac{D}{d}\omega-\frac{\omega^2}{d} \qquad \text{for } n \text{ even} \tag{1.4}$$

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are units of $\mathbb{Q}(\omega)$ (called *truncated units*, because they involve only a few non-zero terms).

Letting

$$\alpha = \tfrac{1}{2}D + \tfrac{1}{2}\sqrt{D^2+4d} \qquad \text{and} \qquad \beta = \tfrac{1}{2}D - \tfrac{1}{2}\sqrt{D^2+4d}, \tag{1.5}$$

we showed in [3] that

$$M_n = \alpha^n + \beta^n \qquad \text{and} \qquad \varepsilon = 1 + \frac{D}{d}\omega - \frac{\omega^2}{d} = \frac{(\omega-\alpha)(\omega-\beta)}{\alpha\beta},$$

this last factorization of ε providing a short proof of the fact that ε is a unit of $\mathbb{Q}(\omega)$.

We know that units of the form $e + x\omega$, the so-called binomial units, are most important in solving Thue equations (over the ground field $\mathbb{Q}$). We just saw that a unit of the form $e + x\omega + y\omega^2$ has a factorization as a product of two binomial units, except that the ground field, instead of being $\mathbb{Q}$, is a quadratic extension $\mathbb{Q}(\sqrt{m})$ of $\mathbb{Q}$. We therefore wonder whether such binomial units lying in a field which is an algebraic extension of a quadratic field $\mathbb{Q}(\sqrt{m})$ might have interesting applications in solving Thue equations over the ground field $\mathbb{Q}(\sqrt{m})$.

In fact, to this author, one of the most interesting applications of units is to bring useful pieces of information in order to master the arithmetic of a given algebraic number field K. The ultimate end is to obtain a fundamental system of units of the field K, since the regulator of the field under consideration will then be known, and since the regulator of K is an important piece of data with which to calculate the class number of K.

The programme a few mathematicians have had in mind for some years is the following; to obtain in a field K:

(1) new units;

(2) an independent system of units;

(3) a maximal independent system of units;

(4) a fundamental system of units.

This programme is well illustrated with the following. First, L. Bernstein and H. Hasse [1] found the unit η in $\mathbb{Q}(\omega)$, where

$$\eta = \frac{(\omega - D)^n}{d}, \qquad \omega = \sqrt[n]{D^n \pm d}, \qquad d \mid D.$$

Then F. Halter-Koch and H. J. Stender [5] obtained an independent system of units in $\mathbb{Q}(\omega)$ with $d \mid D^n$ and remarked that their system of units is maximal for $n = 2$, 3, 4, 6. Finally, in these fields of small degree, H. J. Stender [7] showed that the units considered generate a subgroup of small index in the group of units of $\mathbb{Q}(\omega)$. This programme is also illustrated with the truncated unit found by L. Bernstein [2]. G. Frei and C. Levesque [3] exhibited an independent system of units which happens to be maximal for $n=2$, 3, 4, 6; it turns out that for those small degrees, H. J. Stender [8] calculated the index of the group generated by these independent units in the group of units of $\mathbb{Q}(\omega)$.

We therefore think that the units considered in this paper will lead to other new results along the line of (1), (2), (3), and (4). The referee asked, "Why plunge into such deep, technical formulae and considerations?" In fact, it only illustrates that calculating units is a difficult problem: to prove our results, we use a short cut which, when applied to [2], gives a one-line proof. How complicated would the proof and the notations have become without that idea of factorizing the truncated units under consideration as products of two binomial units? Though we do not have in mind deep applications of our results, we think that time will prove them to be useful. Consider what Sprindžuk did with a maximal independent system of units of a totally real number field of degree n which has been around for many years and which was even rediscovered by different authors: he proved that "almost every" algebraic number field has a large class number [6].

In this paper, we extend Bernstein's result to the field $\mathbb{Q}(\omega)$, where ω is a root of a given irreducible polynomial, and give a necessary and sufficient condition under which $\varepsilon = e + b\omega + c\omega^2$ is a unit of $\mathbb{Q}(\omega)$. Some explicit cases will be investigated: see Theorems 3.1 and 3.6 for the main results. It goes without saying that the results were obtained after extensive calculations; in particular, the cases where $n=3$, 4, 5, 6 were worked out in detail. To give an idea of the extent of our involvement, let us mention that for a field $\mathbb{Q}(\omega)$ of degree n, L. Bernstein [2] considered one specific case (viz., $k_1 = \cdots = k_{n-1} = 0$ in (2.1)) and it will be seen that we are investigating $2^{n-1}n!$ possibilities. At this stage, one can glance at Section 4.

2. Some Units

Let ω be a root of the (assumed) irreducible polynomial

$$f(X) = X^n + k_1 X^{n-1} + \cdots + k_{n-1} X + k_n \tag{2.1}$$

with coefficients in $\mathbb{Z}$. If $\eta_1 = a + b\omega$ with $a, b \in \mathbb{Q}$, $b \neq 0$, and $N(\eta_1)$ denotes the absolute norm $N_{\mathbb{Q}(\omega)/\mathbb{Q}}(\eta_1)$ of η_1, then it is easy to see that

$$N(\eta_1) = (-b)^n f(-a/b),$$

$$N(\eta_1) = a^n - k_1 a^{n-1} b + \cdots + (-1)^{n-1} k_{n-1} a b^{n-1} + (-1)^n k_n b^n, \tag{2.2}$$

so we trivially obtain

PROPOSITION 2.1. *Let $\eta_1 = a + b\omega$ be an algebraic integer of $\mathbb{Q}(\omega)$. Then η_1 is a unit if and only if*

$$(-b)^n f(-a/b) = 1 \quad or \quad -1.$$

Calculating the characteristic polynomial of $\eta_1 = a + b\omega$, we find that η_1 is a root of

$$Z^n + r_1 Z^{n-1} + \cdots + r_{n-1} Z + r_n,$$

where

$$r_i = \sum_{j=0}^{i} \binom{n-j}{i-j} k_j (-a)^{i-j} b^j, \tag{2.3}$$

with $k_0 = 1$ throughout this paper. Therefore $\eta_1 = a + b\omega$ is an algebraic integer if and only if $r_1, r_2, ..., r_n$ are natural integers.

We can also prove

PROPOSITION 2.2. *Let $d = (-b)^n f(-a/b)$ with $a, b \in \mathbb{Z}\backslash\{0\}$, and for $i = 1, ..., n$, let $d \mid k_i$ (whence $d \mid a^n$). Then*

$$\eta = \frac{(a + b\omega)^n}{d}$$

is a unit of $\mathbb{Q}(\omega)$.

Proof. We have $N(\eta) = d^n/d^n = 1$. It remains to prove, as in [4], that η is an algebraic integer. Looking at

$$\eta = \frac{(a + b\omega)^n}{d} = \frac{1}{d} \sum_{i=0}^{n-1} \left(\binom{n}{i} a^{n-i} b^i - k_{n-i} b^n \right) \omega^i,$$

we see that it suffices to show that for $i = 1, ..., n-1$,

$$\psi_i = \frac{1}{d} \binom{n}{i} a^{n-i} b^i \omega^i$$

is an algebraic integer; this is true since

$$\psi_i^n = \binom{n}{i}^n \left(\frac{a^n}{d}\right)^{n-i} b^{ni} \left(\frac{-k_n - k_{n-1}\omega - \cdots - k_1 \omega^{n-1}}{d} \right)^i$$

is the product of a natural integer and an algebraic integer.

Consider the element $\varepsilon = e + b\omega + c\omega^2$, with rational coefficients and $c \neq 0$. We have

$$\varepsilon = c(\omega-\alpha)(\omega-\beta) = \frac{e(\omega-\alpha)(\omega-\beta)}{\alpha\beta} \tag{2.4}$$

with

$$\alpha = -\frac{b}{2c} + \frac{1}{2c}\sqrt{b^2-4ec} \quad \text{and} \quad \beta = -\frac{b}{2c} - \frac{1}{2c}\sqrt{b^2-4ec}, \tag{2.5}$$

where

$$\alpha+\beta = -b/c \quad \text{and} \quad \alpha\beta = e/c. \tag{2.6}$$

For each $r \in \mathbb{N}$, define $M_r(x, y)$ by

$$M_r(x, y) = \sum_{i=0}^{r} \left[\binom{r-1-i}{i-1} + \binom{r-i}{i}\right] x^{r-2i} y^i. \tag{2.7}$$

Here

$$M_{r+2} = xM_{r+1} + yM_r$$

with $M_0(x, y) = 2$ and $M_1(x, y) = x$; moreover

$$\begin{aligned}
M_2(x, y) &= x^2 + 2y,\\
M_3(x, y) &= x^3 + 3xy,\\
M_4(x, y) &= x^4 + 4x^2y + 2y^2,\\
M_5(x, y) &= x^5 + 5x^3y + 5xy^2,\\
M_6(x, y) &= x^6 + 6x^4y + 9x^2y^2 + 2y^3,\\
M_7(x, y) &= x^7 + 7x^5y + 14x^3y^2 + 7xy^3,\\
M_8(x, y) &= x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + 2y^4,\\
M_9(x, y) &= x^9 + 9x^7y + 27x^5y^2 + 30x^3y^3 + 9xy^4.
\end{aligned}$$

As in [3], we have

$$\alpha^r + \beta^r = \frac{M_r(b, -ec)}{(-c)^r} = M_r\left(\frac{b}{-c}, \frac{e}{-c}\right). \tag{2.8}$$

Using (2.4) and (2.2), we find

$$\begin{aligned}
N(\varepsilon) &= c^n N((\omega-\alpha)(\omega-\beta)) = c^n f(\alpha) f(\beta)\\
&= c^n \left(\sum_{i=0}^{n} k_i^2 \alpha^{n-i}\beta^{n-i} + \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} k_i k_{i+j} (\alpha^{n-i}\beta^{n-i-j} + \beta^{n-i}\alpha^{n-i-j}) \right).
\end{aligned}$$

Letting $H=c^n f(\alpha) f(\beta)$ and using (2.6) and (2.8), then in terms of the k_i's, we have

$$H=\sum_{i=0}^{n} k_i^2 e^{n-i} c^i+\sum_{i=0}^{n-1} \sum_{j=1}^{n-i} (-1)^j k_i k_{i+j} e^{n-i-j} c^i M_j(b, -ec). \tag{2.9}$$

We conclude that

$$N(\varepsilon)=H, \tag{2.10}$$

which gives

PROPOSITION 2.3. *Let $\varepsilon=e+b\omega+c\omega^2$ be an algebraic integer of $\mathbb{Q}(\omega)$ and let H be defined as in (2.9). Then ε is a unit if and only if $H=1$ or -1.*

It is worth noting that if $k_1=k_2=\cdots=k_{n-1}=0$, the formula (1.1) obtained by Bernstein is a special case of (2.10).

Now we wish to investigate some cases where ε is a unit. Let us first put

$$e=1 \text{ and } c=-1/d \qquad \text{with} \quad d\in\mathbb{Z}\backslash\{0\},\ b\in\mathbb{Z}. \tag{2.11}$$

Taking the determinant which gives the characteristic polynomial of ω^2/d, adding k_1 times the $(n-1)$st row to the last row, and doing a series of column operations in order to transform the entry $-1/d$ into 0, we find that the characteristic polynomial of ω^2/d is

$$X^{n-2}\begin{vmatrix} \sum_{i=0}^{n} \dfrac{k_{2i}}{d^i X^{i-1}} & \sum_{i=0}^{n} \dfrac{k_{2i+1}}{d^{i+1} X^i} \\ \sum_{i=0}^{n} \dfrac{k_{2i+1}}{d^i X^{i-1}} & \sum_{i=0}^{n} \dfrac{k_{2i}}{d^i X^{i-1}} \end{vmatrix};$$

here $k_0=1$ and $k_j=0$ for $j>n$. Hence a sufficient condition for $\varepsilon=1+b\omega-\omega^2/d$ to be an algebraic integer is

$$d^i \mid k_{2i-1} \text{ and } d^i \mid k_{2i} \qquad \text{for} \quad 1\leqslant 2i-1,\ 2i\leqslant n. \tag{2.12}$$

To conclude this section, we note that the factorization of ε in (2.4) prompts us to take $e=1$ ($b\in\mathbb{Q}$, $c\in\mathbb{Q}\backslash\{0\}$) and to define ξ and λ by

$$\xi=\frac{\omega-\alpha}{\beta} \qquad \text{and} \qquad \lambda=\frac{\omega-\beta}{\alpha}, \tag{2.13}$$

with α and β as in (2.5). If $\varepsilon=1+b\omega+c\omega^2$ is a unit of $\mathbb{Q}(\omega)$ and if ξ (resp. λ) is an algebraic integer, then ξ and λ are units of $L=\mathbb{Q}(\omega, \sqrt{b^2-4c})$ because

$$N_{L/\mathbb{Q}(\omega)}(\xi)=\xi\lambda=\varepsilon, \qquad \text{resp.} \qquad N_{L/\mathbb{Q}(\omega)}(\lambda)=\lambda\xi=\varepsilon. \tag{2.14}$$

Now ξ is a root of $Z^n + s_1 Z^{n-1} + \cdots + s_{n-1} Z + s_n$, with

$$s_i = \sum_{j=0}^{i} \binom{n-j}{i-j} \frac{k_j}{\beta^j} \left(\frac{\alpha}{\beta}\right)^{i-j}. \tag{2.15}$$

Since α/β is a root of $X^2 + (2 - b^2/c)X + 1$, we can conclude that ξ (resp. λ) is an algebraic integer if $b^2/c \in \mathbb{Z}$ and if for $j = 1, ..., n$, the algebraic number k_j/β^j (resp. k_j/α^j) is an algebraic integer.

3. Some Explicit Cases

There are many possibilities for the k_i's of the *irreducible* polynomial of (2.1) for which

$$\varepsilon = 1 + \frac{D}{d}\omega - \frac{\omega^2}{d} \qquad \text{with} \quad D \in \mathbb{N}, d \in \mathbb{Z}, d \mid D \tag{3.1}$$

(here $e = 1$, $c = -1/d$, $D = bd$) inherits the property of being a unit of the field $\mathbb{Q}(\omega)$ of degree n over $\mathbb{Q}$. In this section, we investigate some of these cases.

Assume that $n \geqslant 2$ (we do not lose much in excluding $n = 1$) and that σ is a fixed permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}. \tag{3.2}$$

Moreover suppose that

$$S = \{j_1, j_2, ..., j_s\} \tag{3.3}$$

is a (possibly empty) fixed set of integers such that

$$j_0 = 0 < j_1 < j_2 < \cdots < j_s < n + 1 = j_{s+1}. \tag{3.4}$$

Here by convention $j_0 = 0$ and $j_{s+1} = n + 1$.

We want to prove the following result.

Theorem 3.1. *Let $\sigma^{-1}(n) \notin S$ and let*

$$k_{\sigma(j)} = 0 \qquad \text{for} \quad j \in S = \{j_1, j_2, ..., j_s\}. \tag{3.5}$$

For $r = 0, 1, ..., s$ and for $j_r < j < j_{r+1}$, let

$$k_{\sigma(j)} = (-1)^{j-r} (-d)^{a(j) + (-1)^j c_r} M_{\sigma(j) - 2a(j) - 2(-1)^j c_r}, \tag{3.6}$$

with

$$a(j)=\sum_{t=0}^{j-1}(-1)^t\,\sigma(j-1-t), \tag{3.7}$$

$$c_r=\sum_{i=0}^{r-1}(-1)^{i+j_{r-i}}(a(j_{r-i}+1)-a(j_{r-i})). \tag{3.8}$$

Then the algebraic number

$$\varepsilon=1+\frac{D}{d}\omega-\frac{\omega^2}{d} \qquad (\text{with } d\mid D)$$

is a unit of $\mathbb{Q}(\omega)$. *Moreover*

$$\xi=\frac{\omega-\alpha}{\beta} \qquad \text{and} \qquad \lambda=\frac{\omega-\beta}{\alpha}$$

are units of $L=\mathbb{Q}(\omega, \sqrt{D^2+4d})$.

Remarks. (1) Before starting the proof of this theorem, let us note that $a(1)=0$, $a(2)=\sigma(1)$, $a(3)=\sigma(2)-\sigma(1)=\sigma(2)-a(2)$, ..., $a(j+1)=\sigma(j)-a(j)$, ...

(2) Let us also recall from [3] that the integer M_t, where

$$M_t=M_t(D, d)=\alpha^t+\beta^t, \tag{3.9}$$

with

$$\alpha=\tfrac{1}{2}D+\tfrac{1}{2}\sqrt{D^2+4d}, \qquad \beta=\tfrac{1}{2}D-\tfrac{1}{2}\sqrt{D^2+4d}, \qquad d\mid D,$$

satisfies (for any $r, s\in\mathbb{Z}$) the identities

$$M_{-r}=(-d)^{-r}M_r, \tag{3.10}$$

$$M_rM_s=M_{r+s}+(-d)^s M_{r-s}. \tag{3.11}$$

(3) Let us finally remark that it happens infinitely often that we are dealing with irreducible polynomials: take

$$D=2D_0 \qquad \text{with} \quad D_0 \text{ odd and } d\mid D_0,$$

and apply Eisenstein's criterion with the prime 2,

$$2\mid k_i \qquad \text{but} \quad 4\nmid k_n$$

to obtain irreducibility.

Now ξ and λ are algebraic integers, since by using (2.6) and (2.8) we can show that the conditions stated at the end of Section 2 are satisfied. For instance, $k_{\sigma(j)}/\beta^{\sigma(j)}$ is an algebraic integer, because for $j \notin S$, we obtain from (3.6) and (3.9) that

$$(-1)^{j-r} k_{\sigma(j)}/\beta^{\sigma(j)} = (\alpha/\beta)^{\sigma(j)-a(j)-(-1)^j c_r} + (\alpha/\beta)^{a(j)+(-1)^j c_r},$$

a sum of two algebraic integers.

Moreover, let us remark that using (2.9), (2.10), (2.7), and (3.10), we have

$$\begin{aligned} N(\varepsilon) &= \sum_{i=0}^{n} (-d)^{-i} k_i^2 + \sum_{i=0}^{n-1} \sum_{j=1}^{n-i} (-d)^{-i-j} k_i k_{i+j} M_j(D, d) \\ &= 1 + H_1 + H_2 + \cdots + H_n \end{aligned}$$

with H_j defined by

$$H_j = (-d)^{-\sigma(j)} k_{\sigma(j)} \left(k_{\sigma(j)} + \sum_{t=0}^{j-1} k_{\sigma(t)} M_{\sigma(j)-\sigma(t)}(D, d) \right). \tag{3.12}$$

We conclude that $N(\varepsilon) = 1$ if we have

$$H_j = 0 \qquad \text{for} \quad j = 1, 2, ..., n.$$

Therefore, from now on, we take into account the hypothesis

$$k_{\sigma(j)} = 0 \qquad \text{for} \quad j \in S,$$

and we assume that for $j \notin S$,

$$k_{\sigma(j)} = -\sum_{t=0}^{j-1} k_{\sigma(t)} M_{\sigma(j)-\sigma(t)}. \tag{3.13}$$

In short, for $j \in S$ and for $j \notin S$, we manage always to have $H_j = 0$. It remains only to establish (3.13). The following proposition with $c = \sigma(j)$ then shows that (3.6) and (3.12) match.

PROPOSITION 3.2. *Let $c \in \mathbb{N}$, $r \in \{0, 1, ..., s\}$, $j \in \{1, 2, ..., n\}$ and $j_r < j < j_{r+1}$. Then*

$$-\sum_{t=0}^{j-1} k_{\sigma(t)} N_{c-\sigma(t)} = (-1)^{j-r} (-d)^{a(j)+(-1)^j c_r} M_{c-2a(j)-2(-1)^j c_r}. \tag{3.14}$$

Proof. If we were to give each detail, we would first suppose $r = 0$ and prove (3.14) by induction on j. However, this part is left to the reader since it is similar to the rest of the proof.

Let us assume now that (3.14) holds true for $r-1$. Then for any j such that $j_{r-1}<j<j_r$,

$$-\sum_{t=0}^{j-1} k_{\sigma(t)} M_{c-\sigma(t)} = (-1)^{j-r+1}(-d)^{a(j)+(-1)^j c_{r-1}} M_{c-2a(j)-2(-1)^j c_{r-1}}. \tag{3.15}$$

Let us prove the formula for r. We first prove it for $j=j_r+1$. Remembering that $k_{\sigma(j_r)}=0$ and using (3.15) with $j=j_r-1$, we obtain

$$\begin{aligned}
-\sum_{t=0}^{j_r} k_{\sigma(t)} M_{c-\sigma(t)} &= -\sum_{t=0}^{j_r-1} k_{\sigma(t)} M_{c-\sigma(t)} \\
&= (-1)^{j_r-r+1}(-d)^{a(j_r)+(-1)^{j_r}c_{r-1}} M_{c-2a(j_r)-2(-1)^{j_r}c_{r-1}} \\
&= (-1)^{j_r+1-r}(-d)^{a(j_r+1)+(-1)^{j_r+1}c_r} M_{c-2a(j_r+1)-2(-1)^{j_r+1}c_r},
\end{aligned}$$

since by (3.8),

$$\begin{aligned}
a(j_r+1)+(-1)^{j_r+1}c_r &= a(j_r+1)-(-1)^{j_r}((-1)^{j_r}(a(j_r+1)-a(j_r))-c_{r-1}) \\
&= a(j_r+1)-a(j_r+1)+a(j_r)+(-1)^{j_r}c_{r-1}.
\end{aligned}$$

Now we want to prove that (3.14) holds for $j+1$, by supposing that (3.14) is true for j; then in particular, we suppose that

$$\begin{aligned}
&-\sum_{t=0}^{j-1} k_{\sigma(t)} M_{\sigma(j)-\sigma(t)} \\
&\quad = (-1)^{j-r}(-d)^{a(j)+(-1)^j c_r} M_{\sigma(j)-2a(j)-2(-1)^j c_r} = k_{\sigma(j)}.
\end{aligned}$$

Hence

$$\begin{aligned}
&-\sum_{t=0}^{j} k_{\sigma(t)} M_{c-\sigma(t)} \\
&\quad = \left(-\sum_{t=0}^{j-1} k_{\sigma(t)} M_{c-\sigma(t)}\right) - k_{\sigma(j)} M_{c-\sigma(j)} \\
&\quad = (-1)^{j-r}(-d)^{a(j)+(-1)^j c_r} M_{c-2a(j)-2(-1)^j c_r} \\
&\qquad -(-1)^{j-r}(-d)^{a(j)+(-1)^j c_r} M_{\sigma(j)-2a(j)-2(-1)^j c_r} M_{c-\sigma(j)} \\
&\quad = (-1)^{j-r}(-d)^{a(j)+(-1)^j c_r} \\
&\qquad \times (-(-d)^{\sigma(j)-2a(j)-2(-1)^j c_r} M_{c-2\sigma(j)+2a(j)+2(-1)^j c_r}) \\
&\quad = (-1)^{j+1-r}(-d)^{a(j+1)-(1)^j c_r} M_{c-2a(j+1)+2(-1)^j c_r},
\end{aligned}$$

the last but one equality following from (3.11). This concludes the proof of Proposition 3.2 and secures Theorem 3.1.

For a real number x, let $[x]$ denote as usual the greatest integer contained in x. For $v \in \mathbb{N} \cup \{0\}$, let

$$\langle v \rangle = v - 2[v/2] = \begin{cases} 0 & \text{if } v \text{ is even,} \\ 1 & \text{if } v \text{ is odd.} \end{cases} \tag{3.16}$$

Taking the identity for the permutation σ, we deduce from Theorem 3.1 the following

COROLLARY 3.3. *Let*

$$k_j = 0 \qquad \text{for} \quad j \in S = \{j_1, j_2, ..., j_s\}.$$

For $r = 0, 1, ..., s$ *and for* $j_r < j < j_{r+1}$, *let*

$$k_j = (-1)^{j+r}(-d)^{b(j,r)} M_{g(j,r)}(D, d) \tag{3.17}$$

with

$$\begin{aligned} g(j, r) &= 2f_r + (-1)^r \langle j \rangle, \\ b(j, r) &= \binom{j + \langle r \rangle}{2} - f_r, \\ f_r &= \sum_{i=0}^{r} (-1)^{i+1} \langle j_i \rangle. \end{aligned} \tag{3.18}$$

Then the algebraic number

$$\varepsilon = 1 + \frac{D}{d}\omega - \frac{\omega^2}{d} \qquad (\text{with } d \mid D)$$

is a unit of $\mathbb{Q}(\omega)$. *Moreover*

$$\xi = \frac{\omega - \alpha}{\beta} \qquad \text{and} \qquad \lambda = \frac{\omega - \beta}{\alpha}$$

are units of $L = \mathbb{Q}(\omega, \sqrt{D^2 + 4d})$.

Proof. Here $\sigma = 1$. We easily find that

$$a(j) = [j/2] \qquad \text{and} \qquad a(j+1) - a(j) = \langle j \rangle.$$

Then the integer k_j of Theorem 3.1 is

$$k_j = (-1)^{j-r}(-d)^{[j/2] + (-1)^j c_r} M_{\langle j \rangle - 2(-1)^j c_r},$$

where

$$c_r = (-1)^{j_r}\langle j_r\rangle - (-1)^{j_{r-1}}\langle j_{r-1}\rangle + \cdots + (-1)^{r-1}(-1)^{j_1}\langle j_1\rangle$$
$$= -\langle j_r\rangle + \langle j_{r-1}\rangle - \cdots + (-1)^r\langle j_1\rangle = (-1)^r f_r.$$

To see that this k_j is the same as in (3.17), consider the four individual cases given by the parity of r and j, and when $r+j$ is even, use (3.10).

Taking this time the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ n & n-1 & \cdots & 2 & 1 \end{pmatrix},$$

we deduce another corollary of Theorem 3.1.

COROLLARY 3.4. *Let*

$$k_n = -M_n(D, d) \text{ and } k_{n-j} = 0 \text{ for } j \in T = \{j_1, j_2, ..., j_s\}.$$

For $r = 0, 1, ..., s$ *and for* $j_r < j < j_{r+1}$, *let*

$$k_{n-j} = \frac{(-1)^{j+r+1} M_{f(j,r)}(D, d)}{(-d)^{a(j,r)}} \tag{3.19}$$

with

$$\begin{aligned} f(j, r) &= n + 2f_r + (-1)^r\langle j\rangle, \\ a(j, r) &= \left[\frac{j + \langle r+1\rangle}{2}\right] + f_r, \\ f_r &= \sum_{i=0}^{r} (-1)^{i+1}\langle j_i\rangle. \end{aligned} \tag{3.20}$$

Then the algebraic number

$$\varepsilon = 1 + \frac{D}{d}\omega - \frac{\omega^2}{d} \qquad (\textit{with } d \mid D)$$

is a unit of $\mathbb{Q}(\omega)$. *Moreover,*

$$\xi = \frac{\omega - \alpha}{\beta} \quad \textit{and} \quad \lambda = \frac{\omega - \beta}{\alpha}$$

are units of $L = \mathbb{Q}(\omega, \sqrt{D^2 + 4d})$.

Proof. Here $\sigma(j) = n + 1 - j$, and we easily find

$$a(j) = \langle j+1\rangle(n+1) - [j/2] \text{ and } a(j+1) - a(j) = (-1)^{j+1}n - \langle j+1\rangle.$$

Then the integer $k_{\sigma(j)}$ with respect to the set S of Theorem 3.1 is

$$k_{n+1-j} = (-1)^{j+r}(-d)^u M_{n+1-j-2u}$$

with

$$u = \langle j+1 \rangle (n+1) - [j/2] + (-1)^j c_r,$$

where

$$\begin{aligned} c_r &= (-1)^{j_r}(-n(-1)^{j_r} - \langle j_r+1 \rangle) - (-1)^{j_{r-1}}(-n(-1)^{j_{r-1}} - \langle j_{r-1}+1 \rangle) \\ &\quad + \cdots + (-1)^{r-1}(-1)^{j_1}(-n(-1)^{j_1} - \langle j_1+1 \rangle) \\ &= -\langle r \rangle n - \langle j_r+1 \rangle + \langle j_{r-1}+1 \rangle - \cdots + (-1)^r \langle j_1+1 \rangle. \end{aligned}$$

Now let us make the substitution

$$j \mapsto j+1.$$

Recalling that $k_{\sigma(j_r)} = 0$, i.e., $k_{n-t} = 0$ for $t \in \{j_1 - 1, j_2 - 1, ..., j_s - 1\}$, we have $j_r \mapsto j_r + 1$. Then the last integer k_{n+1-j} becomes

$$k_{n-j} = (-1)^{j+r+1}(-d)^v M_{n-j-2v}$$

with

$$\begin{aligned} v &= \langle j \rangle (n+1) - [(j+1)/2] + (-1)^j \langle r \rangle \\ &\quad - (-1)^{j+r}(\langle j_1 \rangle - \langle j_2 \rangle + \cdots + (-1)^{r-1} \langle j_r \rangle). \end{aligned}$$

It remains only to compare it with the value given by (3.19). The four individual cases given by the parity of r and j are left to the reader as tedious calculations.

Let us say in passing that we preferred to state Corollary 3.4 as it stands with $T = \{j_1, ..., j_s\}$ rather than with $\{j_1 - 1, ..., j_s - 1\}$ in order to show its similarity wih Corollary 3.3.

Letting $F = \mathbb{Q}(\sqrt{D^2 + 4d})$, we can prove the following result, which will be used later to obtain the criterion of Theorem 3.6.

PROPOSITION 3.5. *Let*

$$b = n + a(n) - \sigma(n) + (-1)^n c_s. \tag{3.21}$$

Then

(i) $N_{L/F}(\xi) = (-1)^s (\alpha/\beta)^b$,

(ii) $N_{L/F}(\lambda) = (-1)^s (\beta/\alpha)^b$.

Proof. Using the value of $k_{\sigma(j)}$ in Theorem 3.1, we have

$$\begin{aligned}
&(-1)^n N_{L/F}(\omega-\alpha)\\
&\quad=\sum_{j=0}^{n} k_j\alpha^{n-j}=\sum_{j=0}^{n} k_{\sigma(j)}\alpha^{n-\sigma(j)}\\
&\quad=\left(\alpha^n+\sum_{r=0}^{s}\sum_{j=j_r+1}^{j_{r+1}-1} k_{\sigma(j)}\alpha^{n-\sigma(j)}\right)\\
&\quad=\alpha^n+\sum_{r=0}^{s}\left(\begin{array}{l}\displaystyle\sum_{j=j_r+1}^{j_{r+1}-1}(-1)^{j-r}\alpha^{n-a(j)-(-1)^j c_r}\beta^{a(j)+(-1)^j c_r}\\ \displaystyle\qquad+\sum_{j=j_r+1}^{j_{r+1}-1}(-1)^{j-r}\alpha^{n-a(j+1)+(-1)^j c_r}\beta^{a(j+1)-(-1)^j c_r}\end{array}\right)\\
&\quad=\alpha^n+\sum_{r=0}^{s}\left(\begin{array}{l}(-1)^{j_r+1-r}\alpha^{n-a(j_r+1)+(-1)^{j_r}c_r}\beta^{a(j_r+1)-(-1)^{j_r}c_r}\\ \qquad-(-1)^{j_{r+1}-r}\alpha^{n-a(j_{r+1})-(-1)^{j_{r+1}}c_r}\beta^{a(j_{r+1})+(-1)^{j_{r+1}}c_r}\end{array}\right)\\
&\quad=\alpha^n-\alpha^n+(-1)^{j_{s+1}-s-1}\alpha^{n-a(j_{s+1})-(-1)^{j_{s+1}}c_s}\beta^{a(j_{s+1})+(-1)^{j_{s+1}}c_s}\\
&\qquad-\sum_{r=1}^{s}(-1)^{j_r-r}\alpha^{n-a(j_r+1)+(-1)^{j_r}c_r}\beta^{a(j_r+1)-(-1)^{j_r}c_r}\\
&\qquad+\sum_{r=1}^{s}(-1)^{j_r-r}\alpha^{n-a(j_r)-(-1)^{j_r}c_{r-1}}\beta^{a(j_r)+(-1)^{j_r}c_{r-1}}.
\end{aligned}$$

Replacing in the first sum c_r by the value

$$c_r=(-1)^{j_r}(a(j_r+1)-a(j_r))-c_{r-1},$$

the reader will see that the last two sums give 0. Since $N_{L/F}(\beta)=\beta^n$, we finally obtain

$$N_{L/F}\left(\frac{\omega-\alpha}{\beta}\right)=\left(\frac{\alpha}{\beta}\right)^{n-a(j_{s+1})-(-1)^{j_{s+1}}c_s},$$

whence the conclusion in (i), since $a(j_{s+1})=a(n+1)=\sigma(n)-a(n)$.

By symmetry, we obtain $N_{L/F}(\lambda)$ by replacing α and β in the above formula by β and α, respectively.

We can now complement Theorem 3.1 with the following

THEOREM 3.6. *With the $k_{\sigma(j)}$ defined as in* (3.5) *and* (3.6), *suppose that the integer b of* (3.21) *is even. Then*

$$\varepsilon=1+\frac{D}{d}\omega-\frac{\omega^2}{d}\qquad \textit{and}\qquad \varepsilon'=1-\frac{D}{d}\omega-\frac{\omega^2}{d}\qquad (\textit{with } d\mid D)$$

are units of $\mathbb{Q}(\omega)$. *Moreover*

$$\xi = \frac{\omega - \alpha}{\beta}, \qquad \xi' = \frac{\omega + \alpha}{\beta}, \qquad \lambda = \frac{\omega - \beta}{\alpha}, \qquad \lambda' = \frac{\omega + \beta}{\alpha}$$

are units of $L = \mathbb{Q}(\omega, \sqrt{D^2 + 4d})$.

Proof. Since b is even, we can substitute $-\alpha$ for α in ξ; for the same reason, we can substitute $-\beta$ for β in λ. Since $\xi'\lambda' = \varepsilon'$, we have the conclusion.

It could be shown that there was no loss of generality in taking $e = 1$, at the beginning of this section, but the details are not interesting.

EXAMPLE 1. For $j \in \{0, 1, ..., n-1\}$, we can take

$$k_{n-j} = (-1)^{j+1} M_{n+\langle j \rangle}/(-d)^{(j+\langle j \rangle)/2}.$$

EXAMPLE 2. For a given integer s such that $1 \leqslant s \leqslant n-1$, take

$$k_{n-i} = 0 \quad \text{for} \quad i \notin \{s, n\},$$

$$k_{n-s} = M_{n+s}/(-d)^s \quad \text{and} \quad k_n = -M_n.$$

4. SPECIFIC EXAMPLES

The results of Section 3 were obtained after extensive calculations. Some of the cases which were worked out in detail are considered in this section.

In each of the following examples, we give some values of the k_i's for which the element ε of (3.1) is a unit of $\mathbb{Q}(\omega)$, where ω is a root of the (assumed) irreducible polynomial $f(X)$ of (2.1). Remember that $M_t = M_t(D, d)$ with $d \mid D$, that $M_0 = 2$, and that a permutation σ will be written as a product of cycles. Now the reader can understand that for a given degree n, there are $2^{n-1}n!$ possibilities to consider, i.e., the number of candidates for $S = \{j_1, ..., j_s\}$ times the number of permutations.

In fact, in the last count there are some repetitions. If for a given $i \in \{0, 1, ..., n-1\}$ we have i null coefficients in the polynomial $f(X) = X^n + k_1 X^{n-1} + \cdots + k_{n-1} X + k_n$, then we have $\binom{n-1}{i}$ different possibilities for $f(X)$ to start with; moreover, from the definition (3.6) we have $(n-i)!$ choices for the non-zero coefficients; so we are dealing with at most $g_n = \sum_{i=0}^{n-1} \binom{n-1}{i}(n-i)!$ different cases. One can verify in passing that for $n = 4$ there are indeed 46 disjoint cases among the $g_4 = 49$ cases considered; this could be explained by symmetry reasons.

Cases with $n=4$

k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4
	$\sigma=(1)$				$\sigma=(1\ 2)$		
$-M_1$	$-dM_0$	dM_1	d^2M_0	$-d^{-1}M_3$	$-M_2$	$d^{-1}M_5$	M_4
0	$-M_2$	$-dM_1$	dM_2	0	$-M_2$	$-dM_1$	dM_2
$-M_1$	0	$-dM_1$	$-d^2M_0$	$-M_1$	0	$-dM_1$	$-d^2M_0$
$-M_1$	$-dM_0$	0	dM_2	$-d^{-1}M_3$	$-M_2$	0	$d^{-1}M_6$
0	0	$-M_3$	$-dM_2$	0	0	$-M_3$	$-dM_2$
0	$-M_2$	0	d^2M_0	0	$-M_2$	0	d^2M_0
$-M_1$	0	0	$-dM_2$	$-M_1$	0	0	$-dM_2$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(1\ 3)$				$\sigma=(1\ 4)$		
$d^{-1}M_3$	$-d^{-1}M_4$	$-M_3$	d^2M_0	$d^{-4}M_9$	$d^{-2}M_6$	$-d^{-2}M_7$	$-M_4$
0	$-d^{-1}M_4$	$-M_3$	$d^{-1}M_6$	0	$d^{-2}M_6$	$-d^{-2}M_7$	$-M_4$
$d^{-2}M_5$	0	$-M_3$	$-d^{-2}M_8$	$d^{-1}M_3$	0	$-d^{-1}M_5$	$-M_4$
$-d^{-1}M_3$	$-M_2$	0	$d^{-1}M_6$	$-d^{-2}M_5$	$d^{-2}M_6$	0	$-M_4$
0	0	$-M_3$	$-dM_2$	0	0	$-d^{-1}M_5$	$-M_4$
0	$-M_2$	0	d^2M_0	0	$d^{-2}M_6$	0	$-M_4$
$-M_1$	0	0	$-dM_2$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(2\ 3)$				$\sigma=(2\ 4)$		
$-M_1$	$-M_2$	$-dM_1$	M_4	$-M_1$	M_2	$-M_3$	$-dM_2$
0	$-d^{-1}M_4$	$-M_3$	$d^{-1}M_6$	0	$d^{-1}M_4$	$-d^{-1}M_5$	$-M_4$
$-M_1$	0	$-dM_1$	$-d^2M_0$	$-M_1$	0	$-M_3$	$-dM_2$
$-M_1$	$-dM_0$	0	dM_2	$-M_1$	$d^{-1}M_4$	0	$-dM_2$
0	0	$-M_3$	$-dM_2$	0	0	$-d^{-1}M_5$	$-M_4$
0	$-M_2$	0	d^2M_0	0	$d^{-2}M_6$	0	$-M_4$
$-M_1$	0	0	$-dM_2$	$-M_1$	0	0	$-dM_2$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(3\ 4)$				$\sigma=(1\ 2\ 3)$		
$-M_1$	$-dM_0$	M_3	dM_2	$-M_1$	$-M_2$	$-dM_1$	M_4
0	$-M_2$	dM_1	d^2M_0	0	$-M_2$	$-dM_1$	dM_2
$-M_1$	0	$-M_3$	$-dM_2$	$d^{-2}M_5$	0	$-M_3$	$-d^{-2}M_8$
$-M_1$	$-dM_0$	0	dM_2	$-d^{-1}M_3$	$-M_2$	0	$d^{-1}M_6$
0	0	$-d^{-1}M_5$	$-M_4$	0	0	$-M_3$	$-dM_2$
0	$-M_2$	0	d^2M_0	0	$-M_2$	0	d^2M_0
$-M_1$	0	0	$-dM_2$	$-M_1$	0	0	$-dM_2$
0	0	0	$-M_4$	0	0	0	$-M_4$

Table continued

Cases with $n=4$—*Continued*

k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4
	$\sigma=(1\ 3\ 2)$				$\sigma=(1\ 2\ 4)$		
$d^{-2}M_5$	$-d^{-2}M_6$	$-M_3$	M_4	M_1	$-M_2$	dM_1	d^2M_0
0	$-d^{-1}M_4$	$-M_3$	$d^{-1}M_6$	0	$-M_2$	dM_1	d^2M_0
$d^{-2}M_5$	0	$-M_3$	$-d^{-2}M_8$	$d^{-1}M_3$	0	$-d^{-1}M_5$	$-M_4$
$-M_1$	$-dM_0$	0	dM_2	$d^{-1}M_3$	$-M_2$	0	d^2M_0
0	0	$-M_3$	$-dM_2$	0	0	$-d^{-1}M_5$	$-M_4$
0	$-M_2$	0	d^2M_0	0	$-M_2$	0	d^2M_0
$-M_1$	0	0	$-dM_2$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(1\ 4\ 2)$				$\sigma=(1\ 3\ 4)$		
$-d^{-3}M_7$	$d^{-4}M_{10}$	$d^{-3}M_9$	$-M_4$	$d^{-4}M_9$	$-d^{-1}M_4$	$-M_3$	$d^{-1}M_6$
0	$d^{-1}M_4$	$-d^{-1}M_5$	$-M_4$	0	$-d^{-1}M_4$	$-M_3$	$d^{-1}M_6$
$-d^{-3}M_7$	0	$d^{-3}M_9$	$-M_4$	$-M_1$	0	$-M_3$	$-dM_2$
$-d^{-3}M_7$	$d^{-3}M_8$	0	$-M_4$	$d^{-1}M_3$	$-M_2$	0	d^2M_0
0	0	$-d^{-1}M_5$	$-M_4$	0	0	$-M_3$	$-dM_2$
0	$d^{-2}M_6$	0	$-M_4$	0	$-M_2$	0	d^2M_0
$-d^{-3}M_7$	0	0	$-M_4$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(1\ 4\ 3)$				$\sigma=(2\ 3\ 4)$		
$-d^{-2}M_5$	$d^{-2}M_6$	M_3	$-M_4$	$-M_1$	M_2	$-dM_1$	$-d^2M_0$
0	$d^{-2}M_6$	$-d^{-2}M_7$	$-M_4$	0	dM_0	$-M_3$	$-dM_2$
$-d^{-3}M_7$	0	$d^{-3}M_9$	$-M_4$	$-M_1$	0	$-dM_1$	$-d^2M_0$
$-d^{-2}M_5$	$d^{-2}M_6$	0	$-M_4$	$-M_1$	$d^{-1}M_4$	0	$-dM_2$
0	0	$-d^{-1}M_5$	$-M_4$	0	0	$-M_3$	$-dM_2$
0	$d^{-2}M_6$	0	$-M_4$	0	$d^{-2}M_6$	0	$-M_4$
$-d^{-3}M_7$	0	0	$-M_4$	$-M_1$	0	0	$-dM_2$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(2\ 4\ 3)$				$\sigma=(1\ 4)(2\ 3)$		
$-M_1$	$d^{-1}M_4$	$-d^{-1}M_5$	$-dM_2$	$d^{-2}M_5$	$d^{-1}M_4$	$-d^{-1}M_5$	$-M_4$
0	$d^{-2}M_6$	$-d^{-2}M_7$	$-M_4$	0	$d^{-1}M_4$	$-d^{-1}M_5$	$-M_4$
$-M_1$	0	$-M_3$	$-dM_2$	$d^{-1}M_3$	0	$-d^{-1}M_5$	$-M_4$
$-M_1$	$d^{-1}M_4$	0	$-dM_2$	$-d^{-2}M_5$	$d^{-2}M_6$	0	$-M_4$
0	0	$-d^{-1}M_5$	$-M_4$	0	0	$-d^{-1}M_5$	$-M_4$
0	$d^{-2}M_6$	0	$-M_4$	0	$d^{-2}M_6$	0	$-M_4$
$-M_1$	0	0	$-dM_2$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$

Table continued

Cases with $n=4$—*Continued*

k_1	k_2	k_3	k_4	k_1	k_2	k_3	k_4
	$\sigma=(1\ 2)(3\ 4)$				$\sigma=(1\ 3)(2\ 4)$		
$-d^{-1}M_3$	$-M_2$	$d^{-2}M_7$	$d^{-1}M_6$	$-M_1$	M_2	$-M_3$	$-dM_2$
0	$-M_2$	dM_1	d^2M_0	0	dM_0	$-M_3$	$-dM_2$
$-M_1$	0	$-M_3$	$-dM_2$	$-M_1$	0	$-M_3$	$-dM_2$
$-d^{-1}M_3$	$-M_2$	0	$d^{-1}M_6$	$-d^{-3}M_7$	$d^{-3}M_8$	0	$-M_4$
0	0	$-d^{-1}M_5$	$-M_4$	0	0	$-M_3$	$-dM_2$
0	$-M_2$	0	d^2M_0	0	$d^{-2}M_6$	0	$-M_4$
$-M_1$	0	0	$-dM_2$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(1\ 2\ 3\ 4)$				$\sigma=(1\ 2\ 4\ 3)$		
$d^{-2}M_5$	$-M_2$	$-dM_1$	dM_2	$d^{-1}M_3$	$-M_2$	$-d^{-1}M_5$	d^2M_0
0	$-M_2$	$-dM_1$	dM_2	0	$-M_2$	dM_1	d^2M_0
$-M_1$	0	$-M_3$	$-dM_2$	$-d^{-3}M_7$	0	$d^{-3}M_9$	$-M_4$
$d^{-1}M_3$	$-M_2$	0	d^2M_0	$d^{-1}M_3$	$-M_2$	0	d^2M_0
0	0	$-M_3$	$-dM_2$	0	0	$-d^{-1}M_5$	$-M_4$
0	$-M_2$	0	d^2M_0	0	$-M_2$	0	d^2M_0
$-d^{-3}M_7$	0	0	$-M_4$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(1\ 3\ 2\ 4)$				$\sigma=(1\ 3\ 4\ 2)$		
M_1	dM_0	$-M_3$	$-dM_2$	$d^{-2}M_5$	$d^{-4}M_{10}$	$-M_3$	$-d^{-2}M_8$
0	dM_0	$-M_3$	$-dM_2$	0	dM_0	$-M_3$	$-dM_2$
$-M_1$	0	$-M_3$	$-dM_2$	$d^{-2}M_5$	0	$-M_3$	$-d^{-2}M_8$
$-d^{-2}M_5$	$d^{-2}M_6$	0	$-M_4$	$-M_1$	$d^{-1}M_4$	0	$-dM_2$
0	0	$-M_3$	$-dM_2$	0	0	$-M_3$	$-dM_2$
0	$d^{-2}M_6$	0	$-M_4$	0	$d^{-2}M_6$	0	$-M_4$
$-d^{-3}M_7$	0	0	$-M_4$	$-M_1$	0	0	$-dM_2$
0	0	0	$-M_4$	0	0	0	$-M_4$
	$\sigma=(1\ 4\ 2\ 3)$				$\sigma=(1\ 4\ 3\ 2)$		
$d^{-1}M_3$	M_2	$-d^{-1}M_5$	$-M_4$	$-d^{-3}M_7$	$d^{-3}M_8$	$d^{-2}M_7$	$-M_4$
0	$d^{-1}M_4$	$-d^{-1}M_5$	$-M_4$	0	$d^{-2}M_6$	$-d^{-2}M_7$	$-M_4$
$d^{-1}M_3$	0	$-d^{-1}M_5$	$-M_4$	$-d^{-3}M_7$	0	$d^{-3}M_9$	$-M_4$
$-d^{-3}M_7$	$d^{-3}M_8$	0	$-M_4$	$-d^{-3}M_7$	$d^{-3}M_8$	0	$-M_4$
0	0	$-d^{-1}M_5$	$-M_4$	0	0	$-d^{-1}M_5$	$-M_4$
0	$d^{-2}M_6$	0	$-M_4$	0	$d^{-2}M_6$	0	$-M_4$
$-d^{-3}M_7$	0	0	$-M_4$	$-d^{-3}M_7$	0	0	$-M_4$
0	0	0	$-M_4$	0	0	0	$-M_4$

Cases with $n=6$

k_1	k_2	k_3	k_4	k_5	k_6
		$\sigma=(1)$			
$-M_1$	$-dM_0$	dM_1	d^2M_0	$-d^2M_1$	$-d^3M_0$
0	$-M_2$	$-dM_1$	dM_2	d^2M_1	$-d^2M_2$
$-M_1$	0	$-dM_1$	$-d^2M_0$	d^2M_1	d^3M_0
$-M_1$	$-dM_0$	0	dM_2	d^2M_1	$-d^2M_2$
$-M_1$	$-dM_0$	dM_1	0	d^2M_1	d^3M_0
$-M_1$	$-dM_0$	dM_1	d^2M_0	0	$-d^2M_2$
0	0	$-M_3$	$-dM_2$	dM_3	d^2M_2
0	$-M_2$	0	d^2M_0	$-d^2M_1$	$-d^3M_0$
0	$-M_2$	$-dM_1$	0	dM_3	d^2M_2
0	$-M_2$	$-dM_1$	dM_2	0	$-d^3M_0$
$-M_1$	0	0	$-dM_2$	$-d^2M_1$	d^2M_2
$-M_1$	0	$-dM_1$	0	$-d^2M_1$	$-d^3M_0$
$-M_1$	0	$-dM_1$	$-d^2M_0$	0	d^2M_2
$-M_1$	$-dM_0$	0	0	dM_3	d^2M_2
$-M_1$	$-dM_0$	0	dM_2	0	$-d^3M_0$
$-M_1$	$-dM_0$	dM_1	0	0	d^2M_2
0	0	0	$-M_4$	$-dM_3$	dM_4
0	0	$-M_3$	0	d^2M_1	$-d^2M_2$
0	0	$-M_3$	$-dM_2$	0	dM_4
0	$-M_2$	0	0	d^2M_1	d^3M_0
0	$-M_2$	0	d^2M_0	0	$-d^2M_2$
0	$-M_2$	$-dM_1$	0	0	dM_4
$-M_1$	0	0	0	$-dM_3$	$-d^2M_2$
$-M_1$	0	0	$-dM_2$	0	d^3M_0
$-M_1$	0	$-dM_1$	0	0	$-d^2M_2$
$-M_1$	$-dM_0$	0	0	0	dM_4
0	0	0	0	$-M_5$	$-dM_4$
0	0	0	$-M_4$	0	d^2M_2
0	0	$-M_3$	0	0	$-d^3M_0$
0	$-M_2$	0	0	0	d^2M_2
$-M_1$	0	0	0	0	$-dM_4$
0	0	0	0	0	$-M_6$

Table continued

Cases with $n=6$—*Continued*

k_1	k_2	k_3	k_4	k_5	k_6
		$\sigma=\begin{pmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{pmatrix}$			
$-d^{-3}M_7$	$-d^{-2}M_6$	$d^{-2}M_7$	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
0	$-d^{-2}M_6$	$d^{-2}M_7$	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
$-d^{-2}M_5$	0	$d^{-2}M_7$	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
$d^{-3}M_7$	$-d^{-3}M_8$	0	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
$-d^{-2}M_5$	$d^{-2}M_6$	$d^{-1}M_5$	0	$-d^{-1}M_7$	$-M_6$
$d^{-3}M_7$	$-d^{-3}M_8$	$-d^{-2}M_7$	$d^{-2}M_8$	0	$-M_6$
0	0	$d^{-2}M_7$	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
$d^{-4}M_9$	0	0	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
$d^{-2}M_5$	$d^{-1}M_4$	0	0	$-d^{-1}M_7$	$-M_6$
$d^{-4}M_9$	$d^{-3}M_8$	$-d^{-3}M_9$	0	0	$-M_6$
0	$-d^{-3}M_8$	0	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
0	$d^{-2}M_6$	$d^{-1}M_5$	0	$-d^{-1}M_7$	$-M_6$
0	$-d^{-3}M_8$	$-d^{-2}M_7$	$d^{-2}M_8$	0	$-M_6$
$-d^{-3}M_7$	0	$d^{-1}M_5$	0	$-d^{-1}M_7$	$-M_6$
$d^{-4}M_9$	0	$-d^{-2}M_7$	$d^{-2}M_8$	0	$-M_6$
$-d^{-3}M_7$	$-d^{-2}M_6$	0	$d^{-2}M_8$	0	$-M_6$
0	0	0	$d^{-1}M_6$	$-d^{-1}M_7$	$-M_6$
0	0	$d^{-1}M_5$	0	$-d^{-1}M_7$	$-M_6$
0	0	$-d^{-2}M_7$	$d^{-2}M_8$	0	$-M_6$
0	$d^{-1}M_4$	0	0	$-d^{-1}M_7$	$-M_6$
0	$-d^{-2}M_6$	0	$d^{-2}M_8$	0	$-M_6$
0	$d^{-3}M_8$	$-d^{-3}M_9$	0	0	$-M_6$
$d^{-1}M_3$	0	0	0	$-d^{-1}M_7$	$-M_6$
$-d^{-2}M_5$	0	0	$d^{-2}M_8$	0	$-M_6$
$d^{-3}M_7$	0	$-d^{-3}M_9$	0	0	$-M_6$
$-d^{-4}M_9$	$d^{-4}M_{10}$	0	0	0	$-M_6$
0	0	0	0	$-d^{-1}M_7$	$-M_6$
0	0	0	$d^{-2}M_8$	0	$-M_6$
0	0	$-d^{-3}M_9$	0	0	$-M_6$
0	$d^{-4}M_{10}$	0	0	0	$-M_6$
$-d^{-5}M_{11}$	0	0	0	0	$-M_6$
0	0	0	0	0	$-M_6$

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